

On the Lusternik-Schnirelmann category of a simplicial map

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Abstract

In this paper, we study the Lusternik–Schnirelmann category of a simplicial map between simplicial complexes, generalizing the simplicial category of a complex to that of a map. Several properties of this new invariant are shown, including its relevance to products and fibrations. We relate this category of a map to the classical Lusternik–Schnirelmann category of a map between finite topological spaces. Finally, we show how the simplicial category of a map may be used to define and study a simplicial version of the essential category weight of J. Strom.

Keywords: simplicial Lusternik–Schnirelmann category, category of a map, contiguity, strong homotopy type, essential category weight, finite spaces, order complex

2000 MSC: 55M30, 55U05, 55U10, 06F30

1. Introduction

The Lusternik–Schnirelmann (LS) category of a topological space X , denoted $\text{cat}(X)$, was first defined by Lusternik and Schnirelmann in [10] for the purpose of studying critical points on manifolds. Since its humble inception, it has inspired a wide variety of novel techniques in algebraic topology as well as ascertaining many new and interesting numerical invariants of topological spaces. See, for example, the book [5] devoted to the topic.

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Recently, there has been a growing interest in carrying the LS category into the simplicial or discrete setting [1, 7, 15]. In this paper, we continue the work begun by Fernández-Ternero et al. in [7] and [6] by studying the simplicial LS category of a simplicial map. In addition to continuing the work of these authors, our work extends several results from the classical setting due to Bernstein and Ganea [4], Hardie [8], and Strom. After establishing the necessary background in Section 2, we give the definition and basic properties of the simplicial LS category of a simplicial map in Section 3.1. The rest of Section 3 is devoted to studying the simplicial category of a product map as well as a simplicial fibration.

Finally, we apply our theory in Section 4 by giving a relationship between the simplicial category of a map and the classical category of the corresponding finite space. In addition, we continue to illustrate the phenomena between the simplicial and classical LS category by defining and studying basic properties of the so-called essential category of a map, following the work of Strom [13, 14].

2. Background

In this Section we establish the necessary background that will be used throughout this paper. All simplicial complexes are assumed to be finite and (edge-path) connected, and all maps between simplicial complexes are assumed to be simplicial maps. Following the convention in algebraic topology, with domain K and codomain L understood from context, we will use $*$ to denote the simplicial map $*$: $K \rightarrow L$ that sends everything to some vertex in L .

We begin by defining the simplicial maps and our notion of equivalence. Our references for the fundamentals of simplicial complexes and maps between them is [11] and [2].

Definition 1. *Let K and L be (abstract) simplicial complexes. A subset $U \subseteq K$ which is also a simplicial complex is called a **subcomplex**. A **simplicial map** $f: K \rightarrow L$ is a function from the vertex set of K to the vertex set of L such that if σ is a simplex of K , then $f(\sigma)$ is a simplex of L . Two maps $f, g: K \rightarrow L$ are said to be **contiguous**, denoted $f \sim_c g$, if whenever σ is a simplex in K , $f(\sigma) \cup g(\sigma)$ is a simplex in L . Furthermore, we say that f and g are in the same **contiguity class**, denoted $f \sim g$ if there exists a sequence of simplicial maps $f_0, f_1, \dots, f_m: K \rightarrow L$ such that $f = f_0 \sim_c f_1 \sim_c \dots \sim_c$*

$f_{m-1} \sim_c f_m = g$. If $f \sim *$, we say that f is in the **contiguity class of a vertex**.

Definition 2. *Simplicial complexes K and L are said to have the same **strong homotopy type**, denoted $K \sim L$, if there are simplicial maps $f: K \rightarrow L$ and $g: L \rightarrow K$ such that $g \circ f \sim \text{id}_K$ and $f \circ g \sim \text{id}_L$. In this case, we say that f is a **strong homotopy equivalence**.*

Strong homotopy was defined and studied by Barmak and Minian in [3]. It was then used in [7] to define the notion of the simplicial Lusternik–Schnirelmann category of a simplicial complex.

Definition 3. *Let K be a simplicial complex. A subcomplex $U \subseteq K$ is **categorical** if $i_U \sim *$ where $i_U: U \rightarrow K$ is the inclusion and $*$: $U \rightarrow K$ is the map sending everything in U to a single vertex. The **simplicial Lusternik–Schnirelmann category**, **simplicial LS category**, or **simplicial category of K** , denoted $\text{scat}(K)$, is the least integer n such that K can be covered by categorical subcomplexes U_0, U_1, \dots, U_n . We call U_0, U_1, \dots, U_n a **categorical cover of K** .*

We also have the notion of a simplicial “geometric” category which will play a role in Section 3.2.

Definition 4. *The **simplicial geometric category of K** , denoted $\text{gscat}(K)$, is the least integer $m \geq 0$ such that there exists subcomplexes U_0, U_1, \dots, U_m covering K with $\text{id}_{U_j} \sim *$ for all $0 \leq j \leq m$.*

The following relationship between scat and gscat is immediate.

Proposition 5. *[7, Proposition 4.2] For every simplicial complex K , $\text{scat}(K) \leq \text{gscat}(K)$.*

3. Simplicial category of a map

This Section is devoted to investigating the simplicial category of a simplicial map, our main object of study. We note that many of these results are analogues of known results in the classical case (see the book [5]) as well as extensions of results in [7] and [6] for simplicial complexes.

3.1. Basic properties

We begin with the definition of the simplicial category of a simplicial map.

Definition 6. Let $f: K \rightarrow L$ be a simplicial map. The **simplicial Lusternik–Schnirelmann category, simplicial LS category, or simplicial category of f** , denoted $\text{scat}(f)$, is the least integer n such that K can be covered by subcomplexes U_0, U_1, \dots, U_n with $f|_{U_j} \sim *$ for all $0 \leq j \leq n$. We call U_0, \dots, U_n a **categorical cover of f** .

Immediately, we see that this generalizes the definition of simplicial category of a simplicial complex.

Proposition 7. Let K be a simplicial complex. Then $\text{scat}(\text{id}_K) = \text{scat}(K)$.

Proof Assume that $\text{scat}(\text{id}_K) = n$. Then K can be covered by subcomplexes U_0, U_1, \dots, U_n with $\text{id}_K|_{U_j} \sim *$. Since $\text{id}_K|_{U_j} = i_{U_j}$ and $\text{id}_K|_{U_j} \sim *$, $i_{U_j} \sim *$ so that U_0, U_1, \dots, U_n forms a categorical cover of K . The same reasoning shows that a categorical cover of K yields a categorical cover of id_K . Thus $\text{scat}(\text{id}_K) = \text{scat}(K)$. \square

As in the classical case, the simplicial category of a composition bounds below the simplicial category of either map.

Proposition 8. If $f: K \rightarrow L$ and $g: L \rightarrow M$, then

$$\text{scat}(g \circ f) \leq \min\{\text{scat}(g), \text{scat}(f)\}.$$

Proof Assume that $f: K \rightarrow L$ and $g: L \rightarrow M$. The intention is to show that $\text{scat}(g \circ f) \leq \min\{\text{scat}(g), \text{scat}(f)\}$. We first show that $\text{scat}(g \circ f) \leq \text{scat}(f)$. Write $\text{scat}(f) = n$ so that there exists $U_0, U_1, \dots, U_n \subseteq K$ covering K with $f|_{U_j} \sim *$. We claim that $(g \circ f)|_{U_j} \sim *$. Observe that $(g \circ f)|_{U_j} = g \circ (f|_{U_j}) \sim g \circ * \sim *$. Thus, $\text{scat}(g \circ f) \leq \text{scat}(f) = n$.

Now write $\text{scat}(g) = m$. Then there exists $V_0, V_1, \dots, V_m \subseteq L$ such that $g|_{V_j} \sim *$. Define $U_j := f^{-1}(V_j)$ for all $0 \leq j \leq m$. This clearly covers K . Observe that

$$\begin{array}{ccc}
U_j & \xrightarrow{(g \circ f)|_{U_j}} & M \\
f|_{U_j} \downarrow & \nearrow g|_{V_j} & \\
V_j & &
\end{array}$$

commutes up to contiguity class. Since $g|_{V_j} \sim *$, we have $(g \circ f)|_{U_j} \sim *$. \square

It then easily follows that the simplicial category of a map bounds below the simplicial category of either simplicial complex.

Proposition 9. *Let $f: K \rightarrow L$. Then $\text{scat}(f) \leq \min\{\text{scat}(K), \text{scat}(L)\}$.*

Proof Assume that $f: K \rightarrow L$. The intention is to show that $\text{scat}(f) \leq \min\{\text{scat}(K), \text{scat}(L)\}$. Observe that

$$\text{scat}(f) = \text{scat}(f \circ \text{id}_K) \leq \text{scat}(\text{id}_K) = \text{scat}(K)$$

by Propositions 8 and 7. The same argument shows that $\text{scat}(f) \leq \text{scat}(L)$. Thus $\text{scat}(f) \leq \min\{\text{scat}(K), \text{scat}(L)\}$. \square

Remark 10. *It is easy to construct examples where the above inequalities are strict by letting $f = *$ and K have arbitrarily large category.*

We now show that simplicial category is well-defined up to contiguity class. It is a matter of showing it is well-defined up to contiguity followed by formal induction.

Proposition 11. *Let $f, g: K \rightarrow L$. If $f \sim g$, then $\text{scat}(f) = \text{scat}(g)$.*

Proof Assume without loss of generality that $f \sim_c g$ and let U_0, U_1, \dots, U_n be a categorical covering of g so that $g|_{U_j} \sim *$. By definition of contiguity, $f(\sigma) \cup g(\sigma)$ is a simplex in L , so $f|_{U_j}(\sigma) \cup g|_{U_j}(\sigma)$ is also a simplex in L . Thus, $f|_{U_j} \sim_c g|_{U_j} \sim *$ and $\text{scat}(f) \leq \text{scat}(g)$. The other direction is identical. \square

The following Corollary gives an alternative proof to and generalizes Theorem 4.3 of [7].

Corollary 12. *If $f: K \rightarrow L$ is a strong homotopy equivalence, then $\text{scat}(f) = \text{scat}(K) = \text{scat}(L)$.*

Proof Since f is a strong homotopy equivalence, there exists $g: L \rightarrow K$ such that $g \circ f \sim \text{id}_K$ and $f \circ g \sim \text{id}_L$. We have

$$\begin{aligned} \text{scat}(K) &= \text{scat}(\text{id}_K) && \text{by Proposition 7} \\ &= \text{scat}(g \circ f) && \text{by Proposition 11} \\ &\leq \text{scat}(f) && \text{by Proposition 8} \\ &\leq \text{scat}(K) && \text{by Proposition 9.} \end{aligned}$$

It follows that $\text{scat}(K) = \text{scat}(f)$. The exact same argument shows that $\text{scat}(L) = \text{scat}(f)$. \square

We may also relate the simplicial category of a map to the subspace simplicial category. First, a definition.

Definition 13. Let $A \subseteq K$ be a subcomplex. The **subspace simplicial category of A in K** , denoted $\text{scat}_K(A)$, is the least integer n such that there exists $U_0, U_1, \dots, U_n \subseteq K$ that cover A and each U_i is categorical in K .

Proposition 14. Let $i: A \rightarrow K$ be the inclusion. Then $\text{scat}(i) = \text{scat}_K(A)$.

Proof Assume that $i: A \rightarrow K$ is the inclusion, and write $\text{scat}_K(A) = n$. Then there exists $U_0, \dots, U_n \subseteq K$ covering A with $i|_{U_j} \sim *$ where $i|_{U_j}: U_j \rightarrow K$ is the inclusion. Then $U_0 \cap A, U_1 \cap A, \dots, U_n \cap A$ is a categorical cover of i since $i|_{U_j \cap A} = i|_{U_j} \circ a \sim * \circ a \sim *$ where $a: U_j \cap A \rightarrow U_j$ is the inclusion. Hence $\text{scat}(i) \leq \text{scat}_K(A)$. To see the other direction, observe that if V_j is a member of a categorical cover of i , then $* \sim i|_{V_j} = i|_{V_j}$. \square

3.2. Factorization

In this Section, we show that $\text{scat}(f) \leq n$ may be characterized by factorizing through a simplicial complex K' with $\text{gscat}(K') \leq n$. This is an analogue of the result of Bernstein and Ganea [4, Proposition 1.7] as well as a natural generalization of the fact that a continuous map is nullhomotopic if and only if it factors through the cone. First, an easy characterization of a map in the contiguity class of a vertex.

Lemma 15. Let $f: K \rightarrow L$. Then $\text{scat}(f) = 0 \Leftrightarrow f \sim *$.

Proof Assume that $\text{scat}(f) = 0$. Then there exists only U_0 that covers K i.e. $U_0 = K$. Hence $f = f|_K \sim *$. The backwards direction is Proposition 11. \square

Recall that if K is a simplicial complex, we define the **cone on K** by $CK := \{\sigma, \sigma \cup \{v\} : \sigma \in K\}$ where v is some vertex not in K . It is then easy to see that CK has the strong homotopy type of a vertex, whence $\text{scat}(CK) = 0$.

Lemma 16. *Let $f: K \rightarrow L$. Then $f \sim * \Leftrightarrow f$ factors through CK .*

Proof If $f \sim *$, then f clearly factors through CK . Suppose that f factors through CK so that

$$\begin{array}{ccc} & & CK \\ & \nearrow i & \downarrow h \\ K & \xrightarrow{f} & L \end{array}$$

commutes up to contiguity class. Observe that $h \circ i \sim f$ and according to Proposition 8, $\text{scat}(f) \leq \text{scat}(i)$ while $\text{scat}(i) \leq \text{scat}(CK)$ by Proposition 9. But $\text{scat}(CK) = 0$ as noted above. Thus, $\text{scat}(f) = 0$ and by Lemma 15, $f \sim *$. \square

The following two lemmas are stated without proof.

Lemma 17. (*Simplicial Pasting Lemma*) *Let $U, V \subseteq K$ with $f: U \rightarrow L$ and $g: V \rightarrow L$ be simplicial maps. Suppose that $f(v) = g(v)$ whenever $v \in U \cap V$. Then $f \cup g: U \cup V \rightarrow L$ defined by*

$$f \cup g := \begin{cases} f(u) & \text{if } u \in U \\ g(u) & \text{if } u \in V \end{cases}$$

is a well-defined simplicial map.

Lemma 18. *Let $f_1, f_2: U \rightarrow L$ and $g_1, g_2: V \rightarrow L$ satisfy the conditions of the Simplicial Pasting Lemma so that $f_1 \cup g_1$ and $f_2 \cup g_2$ are well-defined. If $f_1 \sim f_2$ and $g_1 \sim g_2$, then $f_1 \cup g_1 \sim f_2 \cup g_2$.*

We now come to one of the main results of this Section. It will be utilized several times in Section 4.2.

Theorem 19. *Let $f: K \rightarrow L$. Then $\text{scat}(f) \leq n \Leftrightarrow$ there exists a simplicial complex K' with $\text{gscat}(K') \leq n$ such that*

$$\begin{array}{ccc} & & K' \\ & \nearrow & \downarrow \\ K & \xrightarrow{f} & L \end{array}$$

commutes up to contiguity class.

Proof First suppose that $\text{scat}(f) \leq n$. Then there exists $U_0, U_1, \dots, U_n \subseteq K$ such that $f|_{U_j} \sim *$. Define $K' := CU_0 \cup CU_1 \cup \dots \cup CU_n$, where CU_j is the cone on U_j . Clearly $\text{gscat}(K') \leq n$. By Lemma 16, there are simplicial maps $\ell_j: U_j \rightarrow CU_j$ and $g_j: CU_j \rightarrow L$ such that

$$\begin{array}{ccc} & & CU_j \\ & \nearrow \ell_j & \downarrow g_j \\ U_j & \xrightarrow{f|_{U_j}} & L \end{array}$$

commutes up to contiguity class i.e. $f|_{U_j} \sim g_j \circ \ell_j$ for all $0 \leq j \leq n$. Combining all these maps together via Lemma 18, we see that $(\bigcup g_j) \circ (\bigcup \ell_j) = \bigcup (g_j \circ \ell_j) \sim \bigcup f|_{U_j} = f$ so that f factors through K' .

Now suppose that there is a K' with $\text{gscat}(K') \leq n$ and that

$$\begin{array}{ccc} & & K' \\ & \nearrow \ell & \downarrow g \\ K & \xrightarrow{f} & L \end{array}$$

commutes up to contiguity class. The intention is to show that $\text{scat}(f) \leq n$. Observe that

$$\begin{aligned} \text{scat}(f) &= \text{scat}(g \circ \ell) && \text{by Proposition 11} \\ &\leq \text{scat}(g) && \text{by Proposition 8} \\ &\leq \text{scat}(K') && \text{by Proposition 9} \\ &\leq \text{gscat}(K') && \text{by Proposition 5} \\ &\leq n && \text{by hypothesis.} \end{aligned}$$

Thus $\text{scat}(f) \leq n$. □

3.3. Products of maps

Let $f: K \rightarrow M$ and $g: L \rightarrow N$ be simplicial maps. In this Section, we study the simplicial category of the map $f \times g: K \times L \rightarrow M \times N$ on the categorical product satisfying the universal property of products ([9, Definition 4.25]). The simplices of $K \times L$ are defined formally as (σ, τ) for each $\sigma \in K$ and $\tau \in L$ with the relation $(\sigma_1, \tau_1) \subseteq (\sigma_2, \tau_2)$ if and only if $\sigma_1 \subseteq \sigma_2$ and $\tau_1 \subseteq \tau_2$ for $\sigma_1, \sigma_2 \in K$ and $\tau_1, \tau_2 \in L$. Then $(f \times g)(\sigma, \tau) := (f(\sigma), g(\tau))$.

We first compute the simplicial category of some basic maps involving products.

Definition 20. *The **(simplicial) diagonal map on K** , denoted $\Delta: K \rightarrow K \times K$, is defined by $\Delta(\sigma) := (\sigma, \sigma)$ for all $\sigma \in K$. The **(simplicial) projection onto the first coordinate** $p_1: K \times L \rightarrow K$ is defined by $p_1(\sigma, \tau) := \sigma$ for all $(\sigma, \tau) \in K \times L$. The simplicial map $p_2: K \times L \rightarrow L$ is defined similarly.*

Proposition 21. *If $\Delta: K \rightarrow K \times K$ is the diagonal map, then $\text{scat}(\Delta) = \text{scat}(K)$.*

Proof According to Proposition 9, $\text{scat}(\Delta) \leq \min\{\text{scat}(K), \text{scat}(K \times K)\}$ and therefore $\text{scat}(\Delta) \leq \text{scat}(K)$. On the other hand, observe that

$$\begin{aligned} \text{scat}(K) &= \text{scat}(\text{id}_K) && \text{by Proposition 7} \\ &= \text{scat}(p_1 \circ \Delta) \\ &\leq \text{scat}(\Delta) && \text{by Proposition 8} \end{aligned}$$

and hence the result. \square

Proposition 22. *If $p_1: K \times L \rightarrow K$, then $\text{scat}(K) = \text{scat}(p_1)$ (similarly for p_2).*

Proof Use the same estimates as Proposition 21 along with the strictly commutative diagram

$$\begin{array}{ccc} K & \hookrightarrow & K \times L \\ \text{id}_K \downarrow & \swarrow p_1 & \\ K & & \end{array}$$

\square

The main result of this Section requires the following Theorem.

Theorem 23. [6, Theorem 5.14] For simplicial complexes K and L , we have $\text{scat}(K \times L) \leq (\text{scat}(K) + 1)(\text{scat}(L) + 1) - 1$.

Theorem 24. Let $f: K \rightarrow M$ and $g: L \rightarrow N$ be simplicial maps. Then

$$\text{scat}(f \times g) \leq (\text{scat}(f) + 1)(\text{scat}(g) + 1) - 1.$$

Proof Write $\text{scat}(f) = n$ and $\text{scat}(g) = m$. By Theorem 19, there exists K', L' with $\text{gscat}(K') \leq n$ and $\text{gscat}(L') \leq m$ along with lifts ϕ_1, ϕ_2 making the following diagrams commute up to contiguity class:

$$\begin{array}{ccc} & K' & \\ \phi_1 \nearrow & \downarrow \psi_1 & \\ K & \xrightarrow{f} & M \end{array} \quad \begin{array}{ccc} & L' & \\ \phi_2 \nearrow & \downarrow \psi_2 & \\ L & \xrightarrow{g} & N. \end{array}$$

Since $f \sim \psi_1 \circ \phi_1$ and $g \sim \psi_2 \circ \phi_2$, we have $f \times g \sim (\psi_1 \times \psi_2) \circ (\phi_1 \times \phi_2)$.

$$\begin{array}{ccc} K \times L & & \\ \phi_1 \times \phi_2 \downarrow & \searrow f \times g & \\ K' \times L & \xrightarrow{\psi_1 \times \psi_2} & M \times N \end{array}$$

Hence

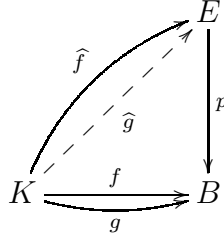
$$\begin{aligned} \text{scat}(f \times g) &= \text{scat}((\psi_1 \times \psi_2) \circ (\phi_1 \times \phi_2)) && \text{by Proposition 11} \\ &\leq \text{scat}(\phi_1 \times \phi_2) && \text{by Proposition 8} \\ &\leq \text{scat}(K' \times L') && \text{by Proposition 9} \\ &\leq (\text{scat}(K') + 1)(\text{scat}(L') + 1) - 1 && \text{by Theorem 23} \\ &\leq (\text{gscat}(K') + 1)(\text{gscat}(L') + 1) - 1 && \text{by Proposition 5} \\ &\leq (n + 1)(m + 1) - 1 \\ &= (\text{scat}(f) + 1)(\text{scat}(g) + 1) - 1. \end{aligned}$$

□

3.4. Fibrations

Fibrations for simplicial complexes were recently defined by Fernández-Ternero et al. [6] and used to prove a result relating the simplicial categories of the complexes in a fibration (see Corollary 27). In this Section, we generalize this result by showing it can easily be deduced from a fact about the simplicial category of maps. We first state the definition of a fibration in our setting.

Definition 25. A simplicial map $E \xrightarrow{p} B$ between path connected simplicial complexes is a **Hurewicz simplicial fibration**, or simply a **fibration**, if for any $f, g: K \rightarrow B$ with $f \sim g$ and for any $\hat{f}: K \rightarrow E$ with $p \circ \hat{f} = f$, there exists a lift $\hat{g}: K \rightarrow E$ such that $\hat{f} \sim \hat{g}$ and $p \circ \hat{g} = g$.



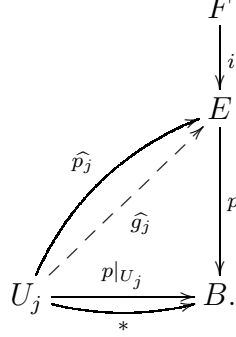
For any $b \in B$, the simplicial complex $F := p^{-1}(\{b\})$ is the **fiber** of p .

It has also been shown by the same authors mentioned above that the fiber is well-defined up to strong homotopy type [6, Theorem 5.2.6]. We now prove a result relating the simplicial categories of maps in a fibration to the simplicial category of the base space. The analogous result in the classical case is originally due to Hardie [8].

Theorem 26. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. Then

$$\text{scat}(E) \leq (\text{scat}(i) + 1)(\text{scat}(p) + 1) - 1.$$

Proof Let $\text{scat}(i) = n$ with cover $V_0, V_1, \dots, V_n \subseteq F$ and $i|_{V_j} \sim *$, and $\text{scat}(p) = m$ with cover $U_0, U_1, \dots, U_m \subseteq E$ and $p|_{U_j} \sim *$. Denoting by $\hat{p}_j: U_j \rightarrow E$ the inclusion, we have the following diagram:



Since $p|_{U_j} \sim *$ and $p \circ \widehat{p}_j = p|_{U_j}$, the fact that p is a fibration implies that there exists a lift $\widehat{g}_j: U_j \rightarrow E$ such that $\widehat{g}_j \sim \widehat{p}_j$ and $p \circ \widehat{g}_j = *$. This latter equation implies that $p(\widehat{g}_j(U_j)) = *$ so $\widehat{g}_j(U_j) \subseteq F$. Define $W_{ij} := \widehat{g}^{-1}(V_j) \subseteq U_j$. Then $\bigcup W_{ij} = E$ since $\bigcup \widehat{g}^{-1}(V_i)$ covers U_j for fixed j and all the U_j cover E . It remains to show that $r_{W_{ij}} \sim *$ where $r_{W_{ij}}: W_{ij} \rightarrow E$ is the inclusion. Let \widehat{g}_{ij} denote the inclusion \widehat{g}_j restricted to W_{ij} into V_j . Then the diagram

$$\begin{array}{ccc}
 W_{ij} & \xrightarrow{r_{W_{ij}}} & E \\
 \widehat{g}_{ij} \downarrow & & \uparrow i \\
 V_j & \xrightarrow{i|_{V_j}} & F
 \end{array}$$

commutes up to contiguity class since $\widehat{g}_j \sim \widehat{p}_j$ with \widehat{p}_j the inclusion. But $i|_{V_j} \sim *$ by hypothesis, whence each W_{ij} is categorical in E , and the result follows. \square

As a Corollary, we immediately obtain the result of Fernández-Ternero et al. relating the scat of the fiber, base space, and total space.

Corollary 27. [6, Theorem 5.2.7] *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. Then $\text{scat}(E) \leq (\text{scat}(F) + 1)(\text{scat}(B) + 1) - 1$.*

Proof By Proposition 8, $\text{scat}(i) \leq \text{scat}(F)$ and $\text{scat}(p) \leq \text{scat}(B)$. Combining this with Theorem 26, we see that $\text{scat}(E) \leq (\text{scat}(i) + 1)(\text{scat}(p) + 1) - 1 \leq (\text{scat}(F) + 1)(\text{scat}(B) + 1) - 1$. \square

4. Applications

4.1. Relationship with finite topological spaces

We briefly look at the relationship between the simplicial category of a map and the classical category of the corresponding continuous map between finite T_0 -spaces and vice versa. Beginning with the work of Stong [12], finite spaces have recently enjoyed a growing interest. Finite T_0 -spaces are in bijective correspondence with posets by declaring that $x \leq y$ if and only if $U_x \subseteq U_y$ where U_x is the intersection of all open sets containing x . Conversely, given a poset (P, \leq) , we define a basis for a topology on P by $U_x := \{y \in P : y \leq x\}$. Furthermore, there is a functor χ from simplicial complexes to posets (finite T_0 -spaces) as well as a functor \mathcal{K} from posets to simplicial complexes.

Let $f: K \rightarrow L$ be a simplicial map. The corresponding poset $\chi(K)$ has vertex set consisting of all the simplices of K while two vertices σ, τ satisfy $\sigma \leq \tau$ if and only if $\sigma \subseteq \tau$. In addition, we obtain a continuous function $\chi(f): \chi(K) \rightarrow \chi(L)$ defined by $\chi(f)(\sigma) := f(\sigma)$. Furthermore, χ takes simplicial maps in the same contiguity class to homotopic maps.

Proposition 28. [2, Proposition 2.1.3] *Let $f, g: K \rightarrow L$ be simplicial maps with $f \sim g$. Then $\chi(f) \simeq \chi(g)$.*

The following Proposition is the analogue for maps found in [7, Proposition 6.4].

Proposition 29. *Let $f: K \rightarrow L$ be a simplicial map. Then $\text{cat}((\chi(f))) \leq \text{scat}(f)$.*

Proof Let $\text{scat}(f) = n$. Then there exists $U_0, U_1, \dots, U_n \subseteq K$ covering K such that $f|_{U_j} \sim *$. By Proposition 28, $\chi(f|_{U_j}) \sim \chi(*) = *$. Observe that $\chi(f|_{U_j})(\sigma) = (f|_{U_j})(\sigma) = \chi(f)|_{\chi(U_j)}(\sigma)$ whence $\chi(U_0), \chi(U_1), \dots, \chi(U_n)$ provides a categorical cover of $\chi(f)$ so that $\text{cat}(\chi(f)) \leq \text{scat}(f)$. \square

We now investigate the other direction. Recall that if X is a finite T_0 -space, we may construct a corresponding simplicial complex, the **order poset** denoted $\mathcal{K}(X)$, by defining the vertex set of $\mathcal{K}(X)$ to be the points of X while simplices are given by the non-empty chains in the order of X or, equivalently, the open sets of X . Furthermore, if $f: X \rightarrow Y$ is continuous, the induced map on the order complexes $\mathcal{K}(f): \mathcal{K}(X) \rightarrow \mathcal{K}(Y)$ is given by

$\mathcal{K}(f)(x) := f(x)$. It can also be shown that if $f, g: X \rightarrow Y$ are continuous maps between finite spaces with $f \simeq g$, then $\mathcal{K}(f) \sim \mathcal{K}(g)$ [2, Proposition 2.12].

Proposition 30. *Let $f: X \rightarrow Y$ be a continuous map between finite topological spaces. Then $\text{scat}(\mathcal{K}(f)) \leq \text{cat}(f)$.*

Proof Let $\text{cat}(f) = n$. Then there are open sets $U_0, U_1, \dots, U_n \subseteq X$ that cover X such that $f|_{U_j} \simeq *$. Then, as mentioned above, $\mathcal{K}(f|_{U_j}) \sim \mathcal{K}(*)$. Since $\mathcal{K}(f|_{U_j})(x) = (f|_{U_j})(x) = \mathcal{K}(f)|_{\mathcal{K}(U_j)}(x)$ and clearly $\mathcal{K}(*) = *$, $\mathcal{K}(U_j)$ forms a categorical cover of $\mathcal{K}(f)$, and the result follows. \square

4.2. Essential simplicial category

The essential category weight of a continuous map was defined by Strom [13, 14] as a means to study the classical category of a space. In this Section, we carry over some of the basic results into the simplicial setting.

Definition 31. *A simplicial map $f: K \rightarrow L$ has **essential simplicial category at least** n , denoted $\text{Es}(f) \geq n$, if for every M with $\text{scat}(M) \leq n$ and simplicial map $g: M \rightarrow K$, we have $f \circ g \sim *$.*

The essential simplicial category is an invariant of the contiguity class of map chosen and furthermore, bounds strictly from below the simplicial category.

Proposition 32. *If $f_0 \sim f_1$, then $\text{Es}(f_0) = \text{Es}(f_1)$. Furthermore, if $f \not\sim *$, then $\text{Es}(f) < \text{scat}(K)$.*

Proof Write $\text{Es}(f_1) = n$. To show that $\text{Es}(f_0) \geq n$, let M be any simplicial complex with $\text{scat}(M) \leq n$ and $g: M \rightarrow K$ be any simplicial map. Since $f_0 \sim f_1$, $f_0 \circ g \sim f_1 \circ g$ and because $\text{Es}(f_1) = n$, $f_1 \circ g \sim *$. Hence $\text{Es}(f_0) \geq n$. The other direction is identical.

Now suppose that $f \not\sim *$, and suppose by contradiction that $\text{Es}(f) \geq \text{scat}(K) = n$. Choosing $M = K$ and $g = \text{id}_K$ as above, we see that $f \sim *$ by definition, a contradiction. \square

Using our characterization of the simplicial category of a map above (Theorem 19), we may in turn give alternative characterizations of the essential simplicial category.

Proposition 33. *Let $f: K \rightarrow L$. The following are equivalent:*

- 1) $\text{Es}(f) \geq n$.
- 2) $f \circ g \sim *$ for all $g: M \rightarrow K$ with $\text{scat}(g) \leq n$.
- 3) $f \circ g \sim *$ for all $g: M \rightarrow K$ with $\text{gscat}(M) \leq n$.

Proof Suppose that $\text{Es}(f) \geq n$ and let $g: M \rightarrow K$ with $\text{scat}(g) \leq n$. Then by Theorem 19, there is an M' with $\text{scat}(M') \leq \text{gscat}(M') \leq n$ and lift making the following diagram commute up to contiguity class:

$$\begin{array}{ccccc} & & M' & & \\ & \nearrow g' & \downarrow h & & \\ M & \xrightarrow{g} & K & \xrightarrow{f} & L. \end{array}$$

Since $\text{Es}(f) \geq n$, $(h \circ g') \circ f \sim *$ so $f \circ g \sim (h \circ g') \circ f \sim *$.

Now assume 2) and let $g: M \rightarrow K$ with $\text{gscat}(M) \leq n$. Applying Theorem 19 to the diagram

$$\begin{array}{ccccc} & & M & & \\ & \nearrow \text{id}_M & \downarrow g & & \\ M & \xrightarrow{g} & K & \xrightarrow{f} & L \end{array}$$

we see that $\text{scat}(g) \leq n$. By 2), $f \circ g \sim *$.

Finally, suppose that 3) holds, and let $g: M \rightarrow K$ with $\text{scat}(M) \leq n$. By Proposition 9, $\text{scat}(g) \leq \text{scat}(M) \leq n$ so by Theorem 19, g factors through a simplicial complex M' with $\text{gscat}(M') \leq n$. By 3), $f \circ g \sim *$ and $\text{Es}(f) \geq n$. \square

Finally, the essential simplicial category of a composition bounds above the product of the essential simplicial category of each of the maps.

Proposition 34. *[14, Cf. Theorem 9] Let $f: K \rightarrow L$ and $g: L \rightarrow M$. Then $\text{Es}(g \circ f) \geq (\text{Es}(g) + 1)(\text{Es}(f) + 1) - 1$.*

Proof Write $\text{Es}(f) = p$ and $\text{Es}(g) = q$ and let $h: Z \rightarrow M$ be any map with $\text{scat}(Z) \leq (p+1) \cdot (q+1) - 1$. Write $Z = U_0 \cup U_1 \cup \dots \cup U_q$ with $\text{scat}(U_i) \leq p$. Since $\text{Es}(f) = p$, we have $(f \circ h)|_{U_j} \sim *$ and

$$\begin{array}{ccccccc} & & & & Z' & & \\ & & & & \downarrow i & & \\ Z & \xrightarrow{h} & K & \xrightarrow{f} & L & \xrightarrow{g} & M \end{array}$$

where $Z' = Z \cup (\amalg U_i)$. Clearly $\text{scat}(Z') \leq q$ and since $\text{Es}(g) = q$, $g \circ i \sim *$, whence $(g \circ f) \circ h \sim *$. We conclude that $\text{Es}(g \circ f) \geq (\text{Es}(g) + 1)(\text{Es}(f) + 1) - 1$. \square

The following is then immediate.

Corollary 35. $\text{Es}(g \circ f) \geq \text{Es}(f)$ and $\text{Es}(g \circ f) \geq \text{Es}(g)$.

5. Future directions

The simplicial category of a simplicial map is a natural generalization of the recently defined simplicial category of a simplicial complex. There are many directions that the work in this paper, and simplicial LS category in general, may go.

One important use of the category of a map was in studying special cases and examples which pertained to the Ganea conjecture. Could the simplicial category of a map be used to study a simplicial version of the Ganea conjecture and related questions? Currently, because different triangulations of spheres may have different strong homotopy type, we do not even know the simplicial category of all simplicial complexes homeomorphic to a sphere.

In [6], the authors give a Whitehead definition of simplicial category in terms of factoring through a simplicial fat wedge. While it was shown that this bounds above the simplicial category, the two strong homotopy invariants, unlike the classical case, are not in general equal. One could give a Whitehead definition of the simplicial category of a map and study it as well as its relationship to the simplicial category of a map defined in this paper.

We have only scratched the surface of the use of the essential simplicial category. For example, given a complex K , can one construct maps f_K such that $\text{scat}(K) = \text{Es}(f_K) + 1$? Strom has used cohomology to produce relations with essential category in the smooth case. How can cohomology provide information to estimate and compute simplicial category?

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